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# On a Classification of Irreducible Almost Commutative Geometries, A Second Helping

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## Abstract

We complete the classification of almost commutative geometries from a particle physics point of view given in [1]. Four missing Krajewski diagrams will be presented after a short introduction into irreducible, non-degenerate spectral triples.

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# 1 Introduction

Alain Connes' noncommutative geometry [2–5] allows in an elegant way to unify gravity and the standard model of particle physics. A central role in this formalism is played by almost commutative spectral triples  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  which decompose into an external and an internal, finite dimensional component. The external part encodes a compact 4-dimensional Euclidian spacetime and the internal one corresponds to a discrete 0-dimensional Kaluza-Klein space, determining the particle content of the theory. Via the spectral action [6] one recovers the Einstein-Hilbert action combined with the bosonic action of a Yang-Mills-Higgs (YMH) theory. Since the set of allowed YMH-theories is determined by the possible internal, finite dimensional spectral triples, we will restrict ourselves to this part. The standard model of particle physics is the most prominent example in this context.

Real, finite dimensional spectral triples have been completely classified by Krajewski [7] and Paschke & Sitarz [8]. A classification of almost commutative geometries from a physical point of view was given in [1]. The spectral triples were required to be irreducible and non-degenerate, in the sense that the Hilbert space was chosen to be as small as possible with non-degenerate fermion masses. Heavy use was made of Krajewski's diagrammatic method, which will be described briefly below. The main obstacle in finding all physically relevant almost commutative spectral triples is the sheer mass of diagrams which have to be considered. Since this is a purely combinatorial problem it is convenient to let a computer do the tedious task. The cases of one and two matrix algebras can still be done by hand. But already three algebras produce hundreds of diagrams and one easily loses sight.

Therefore we developed an algorithm to calculate these diagrams and used the known results from [1] to test and calibrate the program. The main goal was to extend the calculations to more than three algebras, where we expect thousands of possible irreducible spectral triples. During the calibration it turned out that four diagrams were overlooked in the case of three algebras. To complete the proof we will present these four missing diagrams and their models in this paper. The algorithm to compute the diagrams and the results for the case of four algebras will be presented elsewhere.

In section 2 and 3 we will briefly introduce Krajewski diagrams and irreducible, non-degenerate spectral triples. The missing diagrams for three algebras will be presented in section 4.

## 2 Basic definitions and Krajewski diagrams

In this section we will give the necessary basic definitions for a classification of almost commutative geometries from a particle physics point of view. As mentioned above only the 0-dimensional part will be taken into account, so we restrict ourselves to real,  $S^0$ -real, finite spectral triples  $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \epsilon, \chi)$ . The algebra  $\mathcal{A}$  is a finite sum of matrix algebras  $\mathcal{A} = \oplus_{i=1}^N M_{n_i}(\mathbb{K}_i)$  with  $\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$  where  $\mathbb{H}$  denotes the quaternions. A faithful representation  $\rho$  of  $\mathcal{A}$  is given on the finite dimensional Hilbert space  $\mathcal{H}$ . The Dirac operator  $\mathcal{D}$  is a selfadjoint operator on  $\mathcal{H}$  and plays the role of the fermionic mass

matrix.  $J$  is an antiunitary involution,  $J^2 = 1$ , and is interpreted as the charge conjugation operator of particle physics. The  $S^0$ -real structure  $\epsilon$  is a unitary involution,  $\epsilon^2 = 1$ . Its eigenstates with eigenvalue  $+1$  are the particle states, eigenvalue  $-1$  indicates antiparticle states. The chirality  $\chi$  is as well a unitary involution,  $\chi^2 = 1$ , whose eigenstates with eigenvalue  $+1$  ( $-1$ ) are interpreted as right (left) particle states. These operators are required to fulfill Connes' axioms for spectral triples:

- $[J, \mathcal{D}] = [J, \chi] = [\epsilon, \chi] = [\epsilon, \mathcal{D}] = 0$ ,  $\epsilon J = -J\epsilon$ ,  $\mathcal{D}\chi = -\chi\mathcal{D}$ ,  
 $[\chi, \rho(a)] = [\epsilon, \rho(a)] = [\rho(a), J\rho(b)J^{-1}] = [[\mathcal{D}, \rho(a)], J\rho(b)J^{-1}] = 0, \forall a, b \in \mathcal{A}$ .
- The chirality can be written as a finite sum  $\chi = \sum_i \rho(a_i)J\rho(b_i)J^{-1}$ . This condition is called *orientability*.
- The intersection form  $\cap_{ij} := \text{tr}(\chi \rho(p_i)J\rho(p_j)J^{-1})$  is non-degenerate,  $\det \cap \neq 0$ . The  $p_i$  are minimal rank projections in  $\mathcal{A}$ . This condition is called *Poincaré duality*.

Now the Hilbert space  $\mathcal{H}$  and the representation  $\rho$  decompose with respect to the eigenvalues of  $\epsilon$  and  $\chi$  into left and right, particle and antiparticle spinors and representations:

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c \quad (2.1)$$

$$\rho = \rho_L \oplus \rho_R \oplus \overline{\rho}_L^c \oplus \overline{\rho}_R^c \quad (2.2)$$

In this representation the Dirac operator has the form

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathcal{M}} \\ 0 & 0 & \overline{\mathcal{M}}^* & 0 \end{pmatrix}, \quad (2.3)$$

where  $\mathcal{M}$  is the fermionic mass matrix connecting the left and the right handed Fermions.

Since the individual matrix algebras have only one fundamental representation for  $\mathbb{K} = \mathbb{R}, \mathbb{H}$  and two for  $\mathbb{K} = \mathbb{C}$  (the fundamental one and its complex conjugate),  $\rho$  may be written as a direct sum of these fundamental representations with multiplicities

$$\rho(\oplus_{i=1}^N a_i) := (\oplus_{i,j=1; \alpha_i, \alpha_j}^N a_{i\alpha_i} \otimes 1_{m_{j\alpha_j i\alpha_i}} \otimes 1_{(n_j)}) \oplus (\oplus_{i,j=1}^N 1_{(n_i)} \otimes 1_{m_{j\alpha_j i\alpha_i}} \otimes \overline{a_{j\alpha_j}}). \quad (2.4)$$

The first summand denotes the particle sector and the second the antiparticle sector. For the dimensions of the unity matrices we have  $(n) = n$  for  $\mathbb{K} = \mathbb{R}, \mathbb{C}$  and  $(n) = 2n$  for  $\mathbb{K} = \mathbb{H}$  and the convention  $1_0 = 0$ . The index  $\alpha$  indicates whether the representation is the fundamental one or its complex conjugate and thus applies only to the  $\mathbb{K} = \mathbb{C}$  case. The multiplicities  $m_{j\alpha_j i\alpha_i}$  are non-negative integers. Acting with the real structure  $J$  on  $\rho$  permutes the main summands and complex conjugates them. It is also possible to write the chirality as a direct sum

$$\chi = (\oplus_{i,j=1}^N 1_{(n_i)} \otimes \chi_{j\alpha_j i\alpha_i} 1_{m_{j\alpha_j i\alpha_i}} \otimes 1_{(n_j)}) \oplus (\oplus_{i,j=1}^N 1_{(n_i)} \otimes \chi_{j\alpha_j i\alpha_i} 1_{m_{j\alpha_j i\alpha_i}} \otimes 1_{(n_j)}), \quad (2.5)$$

where  $\chi_{j\alpha_j i\alpha_i} = \pm 1$  according to our previous convention on left-(right-)handed spinors. One can now define the multiplicity matrix  $\mu \in M_N(\mathbb{Z})$  such that  $\mu_{j\alpha_j i\alpha_i} := \chi_{j\alpha_j i\alpha_i} m_{j\alpha_j i\alpha_i}$ . This matrix is symmetric and decomposes into a particle and an antiparticle matrix, the second being just the particle matrix transposed,  $\mu = \mu_P + \mu_A = \mu_P + \mu_P^T$ . The entries of the multiplicity matrix must fulfill certain consistency conditions, which are given by table 1 in [7]. It is also possible to recover the intersection form of the Poincaré duality, up to a numerical factor, if the multiplicity matrix is contracted by summation over the  $\alpha$ 's.

The mass matrix  $\mathcal{M}$  of the Dirac operator connects the left and the right handed Fermions. Using the decomposition of the representation  $\rho$  and the corresponding decomposition of the Hilbert space  $\mathcal{H}$  we find two types of submatrices in  $\mathcal{M}$ , namely  $M \otimes 1_{(n_k)}$  and  $1_{(n_k)} \otimes M$ .  $M$  is a complex  $(n_i) \times (n_j)$  matrix connecting the  $i$ -th and the  $j$ -th sub-Hilbert space and its eigenvalues give the masses of the fermion multiplet. We will call the  $k$ -th algebra the colour algebra.

Connes' axioms, the decomposition of the Hilbert space, the representation and the Dirac operator allow a diagrammatic dipiction. As was shown in [7] and [1] this can be boiled down to simple arrows, which encode the multiplicity matrix and the fermionic mass matrix. From these information all the ingredients of the spectral triple can be recovered. For our purpose a simple arrow and a double arrow are sufficient. The arrows allways point from right fermions (positive chirality) to left fermions (negative chirality). We may also restrict ourselves to the particle part, since the information of the antiparticle part is included by transposing the particle part. As an example we will retranslate these two generic arrows into the multiplicity matrix, the representation of the algebra and the Dirac operator. We will adopt the conventions of [1] so that algebra elements tensorised with  $1_{m_{ij}}$  will be written as a direct sum of  $m_{ij}$  summands.

Take the algebra  $\mathcal{A} = \mathbb{H} \oplus M_3(\mathbb{C}) \ni (a, b)$  with the first diagram of Figure 1.

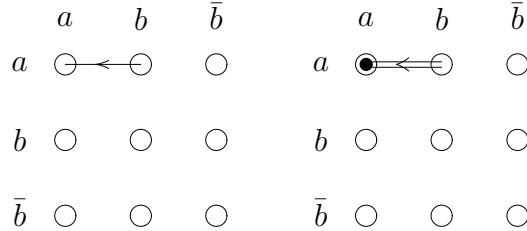


Fig. 1

Then the multiplicity matrix and its contraction are

$$\mu = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mu} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using (2.2), its representation is, up to unitary equivalence

$$\rho_L(a, b) = a \otimes 1_2, \quad \rho_R(a, b) = b \otimes 1_2, \quad \rho_L^c(a, b) = 1_2 \otimes a, \quad \rho_R^c(a, b) = 1_3 \otimes a.$$

The Hilbert space is

$$\mathcal{H} = \mathbb{C}^4 \oplus \mathbb{C}^6 \oplus \mathbb{C}^4 \oplus \mathbb{C}^6.$$

In its Dirac operator (2.3) the mass matrix is  $\mathcal{M} = M \otimes 1_2$ , where  $M$  is a nonvanishing complex  $2 \times 3$  matrix.

Real structure,  $S^0$ -real structure and chirality are given by (cc stands for complex conjugation)

$$J = \begin{pmatrix} 0 & 1_{10} \\ 1_{10} & 0 \end{pmatrix} \circ \text{cc}, \quad \epsilon = \begin{pmatrix} 1_{10} & 0 \\ 0 & -1_{10} \end{pmatrix}, \quad \chi = \begin{pmatrix} -1_4 & 0 & 0 & 0 \\ 0 & 1_6 & 0 & 0 \\ 0 & 0 & -1_4 & 0 \\ 0 & 0 & 0 & 1_6 \end{pmatrix}.$$

The first tensor factor in  $a \otimes 1_2$  concerns particles, the second concerns antiparticles denoted by  $\cdot^c$ . The antiparticle representation is read from the transposed multiplicity matrix.

The second diagram of Figure 1 yields

$$\mu = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mu} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix},$$

and its spectral triple reads:

$$\rho_L(a, b) = a \otimes 1_2, \quad \rho_R(a, b) = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \otimes 1_2,$$

$$\rho_L^c(a, b) = 1_2 \otimes a, \quad \rho_R^c(a, b) = \begin{pmatrix} 1_3 & 0 \\ 0 & 1_3 \end{pmatrix} \otimes a,$$

$$\mathcal{M} = \begin{pmatrix} M_1 & M_2 \end{pmatrix} \otimes 1_2, \quad M_1 \text{ and } M_2 \text{ of size } 2 \times 3,$$

$$J = \begin{pmatrix} 0 & 1_{16} \\ 1_{16} & 0 \end{pmatrix} \circ \text{cc}, \quad \epsilon = \begin{pmatrix} 1_{16} & 0 \\ 0 & -1_{16} \end{pmatrix}, \quad \chi = \begin{pmatrix} -1_4 & 0 & 0 & 0 \\ 0 & 1_{12} & 0 & 0 \\ 0 & 0 & -1_4 & 0 \\ 0 & 0 & 0 & 1_{12} \end{pmatrix}.$$

It should be clear that the number of possible ways to fit one and more arrows into a diagram increases factorially with the number of matrix algebras, i.e. with the size of the diagram. Thus this seemingly simple problem soon becomes intractable, even for a powerful computer.

We started out with the flat Dirac operator of a 4-dimensional spacetime with a fixed fermionic mass matrix. To generate curvature we have to perform a general coordinate transformation and then fluctuate the Dirac operator. This can be achieved by lifting the automorphisms of the algebra to the Hilbert space, unitarily transforming the Dirac operator with the lifted automorphisms and then building linear combinations. Again we restrict ourselves to the finite case. Except for complex conjugation in  $M_n(\mathbb{C})$  and

permutations of identical summands in the algebra  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots \oplus \mathcal{A}_N$ , every algebra automorphism  $\sigma$  is inner,  $\sigma(a) = uau^{-1}$  for a unitary  $u \in U(\mathcal{A})$ . Therefore the connected component of the automorphism group is  $\text{Aut}(\mathcal{A})^e = U(\mathcal{A})/(U(\mathcal{A}) \cap \text{Center}(\mathcal{A}))$ . Its lift to the Hilbert space [4]

$$L(\sigma) = \rho(u)J\rho(u)J^{-1}$$

is multi-valued.

The *fluctuation*  ${}^f\mathcal{D}$  of the Dirac operator  $\mathcal{D}$  is given by a finite collection  $f$  of real numbers  $r_j$  and algebra automorphisms  $\sigma_j \in \text{Aut}(\mathcal{A})^e$  such that

$${}^f\mathcal{D} := \sum_j r_j L(\sigma_j) \mathcal{D} L(\sigma_j)^{-1}, \quad r_j \in \mathbb{R}, \quad \sigma_j \in \text{Aut}(\mathcal{A})^e.$$

The fluctuated Dirac operator  ${}^f\mathcal{D}$  is often denoted by  $\varphi$ , the ‘Higgs scalar’, in the physics literature. We consider only fluctuations with real coefficients since  ${}^f\mathcal{D}$  must remain selfadjoint.

To avoid the multi-valuedness in the fluctuations, we allow the entire unitary group viewed as a (maximal) central extension of the automorphism group.

An almost commutative geometry is the tensor product of a finite noncommutative triple with an infinite, commutative spectral triple. By Connes’ reconstruction theorem [5] we know that the latter comes from a Riemannian spin manifold, which we will take to be any 4-dimensional, compact, flat manifold like the flat 4-torus. The spectral action of this almost commutative spectral triple reduced to the finite part is a functional on the vector space of all fluctuated, finite Dirac operators:

$$V({}^f\mathcal{D}) = \lambda \, \text{tr}[({}^f\mathcal{D})^4] - \frac{\mu^2}{2} \, \text{tr}[({}^f\mathcal{D})^2],$$

where  $\lambda$  and  $\mu$  are positive constants [2, 9]. The spectral action is invariant under lifted automorphisms and even under the unitary group  $U(\mathcal{A}) \ni u$ ,

$$V([\rho(u)J\rho(u)J^{-1}] {}^f\mathcal{D} [\rho(u)J\rho(u)J^{-1}]^{-1}) = V({}^f\mathcal{D}),$$

and it is bounded from below. Our task is to find the minima  $\hat{{}^f\mathcal{D}}$  of this action and their spectra.

### 3 Irreducibility, Non-Degeneracy

To classify the almost commutative spectral triples we will impose some extra conditions as in [1]. We will require the spectral triples to be irreducible and non-degenerate according to the following definitions:

**Definition 3.1.** i) A spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is *degenerate* if the kernel of  $\mathcal{D}$  contains a non-trivial subspace of the complex Hilbert space  $\mathcal{H}$  invariant under the representation  $\rho$  on  $\mathcal{H}$  of the real algebra  $\mathcal{A}$ .

ii) A non-degenerate spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is *reducible* if there is a proper subspace

$\mathcal{H}_0 \subset \mathcal{H}$  invariant under the algebra  $\rho(\mathcal{A})$  such that  $(\mathcal{A}, \mathcal{H}_0, \mathcal{D}|_{\mathcal{H}_0})$  is a non-degenerate spectral triple. If the triple is real,  $S^0$ -real and even, we require the subspace  $\mathcal{H}_0$  to be also invariant under the real structure  $J$ , the  $S^0$ -real structure  $\epsilon$  and under the chirality  $\chi$  such that the triple  $(\mathcal{A}, \mathcal{H}_0, \mathcal{D}|_{\mathcal{H}_0})$  is again real,  $S^0$ -real and even.

**Definition 3.2.** The irreducible spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is *dynamically non-degenerate* if all minima  $\hat{f}\mathcal{D}$  of the action  $V(\hat{f}\mathcal{D})$  define a non-degenerate spectral triple  $(\mathcal{A}, \mathcal{H}, \hat{f}\mathcal{D})$  and if the spectra of all minima have no degeneracies other than the three kinematical degeneracies: left-right, particle-antiparticle and colour. Of course in the massless case there is no left-right degeneracy. We also suppose that the colour degeneracies are protected by the little group. By this we mean that all eigenvectors of  $\hat{f}\mathcal{D}$  corresponding to the same eigenvalue are in a common orbit of the little group (and scalar multiplication and charge conjugation).

In physicists' language non-degeneracy excludes all models with pairwise equal fermion masses in the left handed particle sector up to colour degeneracy. Irreducibility tells us that the Krajewski diagrams, which we have to find must not contain more arrows than strictly necessary to satisfy Connes' Axioms, especially the Poincaré duality. The last requirement of definition 3.2 means noncommutative colour groups are unbroken. It ensures that the corresponding mass degeneracies are protected from quantum corrections.

## 4 The Missing Diagrams

In this section we will present the diagrams missing in the proof for three algebras in [1]. For every diagram only one representative model will be given. All the other models can be obtained by simply exchanging left with right and particles with antiparticles. On the diagrammatic side this is equivalent to changing the directions of all arrows or reflecting the diagram on its diagonal. Permutations of the diagrams are neglected as well, since they lead to the same physical models with a different order of the particles. For every diagram there are several ways to connect the algebras by arrows in accordance with the consistency conditions of table 1. in [7]. With respect to this, the four diagrams are all computed in the same way and they all fall in the same way. The possibilities of complex conjugating an algebra representation are limited and yield no essentially new models. It should be obvious from the diagrams whether the matrix algebras are complex, real or quaternionic. In all other cases the choice of the field will not affect the calculations, so we will not specify the algebras explicitly. For the four missing diagrams,  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  denote the algebras,  $a, b, c$  their generic elements and  $k, \ell, p$  the respective size of the matrices.

**Diagram 1** yields the representation

$$\rho_L(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 & 0 \\ 0 & c \otimes 1_k & 0 \\ 0 & 0 & b \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} \bar{a} \otimes 1_k & 0 & 0 \\ 0 & \bar{b} \otimes 1_k & 0 \\ 0 & 0 & a \otimes 1_p \end{pmatrix},$$

$$\rho_L^c(a, b, c) = \begin{pmatrix} 1_\ell \otimes a & 0 & 0 \\ 0 & 1_p \otimes a & 0 \\ 0 & 0 & 1_\ell \otimes \bar{c} \end{pmatrix}, \quad \rho_R^c(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 & 0 \\ 0 & 1_\ell \otimes a & 0 \\ 0 & 0 & 1_p \otimes \bar{c} \end{pmatrix}.$$

The mass matrix is

$$\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 & 0 \\ 0 & M_2 \otimes 1_k & 0 \\ 0 & 0 & M_3 \otimes 1_p \end{pmatrix}, \quad M_1, M_3 \in M_{k \times \ell}(\mathbb{C}), \quad M_2 \in M_{p \times \ell}(\mathbb{C}),$$

where all three algebras are  $M_n(\mathbb{C})$ . The fluctuations are

$$\begin{aligned} {}^f M_1 &= \sum_j r_j v_j M_1 \bar{u}_j^{-1}, \quad u_j \in U(\mathcal{A}_1), \quad v_j \in U(\mathcal{A}_2), \\ {}^f M_2 &= \sum_j r_j w_j M_2 \bar{v}_j^{-1}, \quad w_j \in U(\mathcal{A}_3), \\ {}^f M_3 &= \sum_j r_j v_j M_3 u_j^{-1}, \end{aligned}$$

and the action  $V(C_1, C_2, C_3)$  is, with  $C_i := {}^f M_i^* {}^f M_i$  equal to

$$4k [\lambda \operatorname{tr}(C_1)^2 - \frac{1}{2} \mu^2 \operatorname{tr}(C_1)] + 4k [\lambda \operatorname{tr}(C_2)^2 - \frac{1}{2} \mu^2 \operatorname{tr}(C_2)] + 4p [\lambda \operatorname{tr}(C_3)^2 - \frac{1}{2} \mu^2 \operatorname{tr}(C_3)].$$

Counting neutrinos and imposing broken colour to be commutative leaves only one case,  $k = \ell = p = 1$ . The fluctuations decouple the  ${}^f M_i$  so it is always possible to reach the absolute minimum of the Higgs potential and the triple is degenerate.

**Diagram 2** falls in the same way.

**Diagram 3** is degenerate in the commutative case and exhibits mass relations in the noncommutative case. The calculation runs along the lines of diagram 8 in [1].

**Diagram 4** yields the representation

$$\begin{aligned} \rho_L(a, b, c) &= \begin{pmatrix} c \otimes 1_k & 0 \\ 0 & a \otimes 1_\ell \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} \bar{b} \otimes 1_k & 0 & 0 \\ 0 & \bar{b} \otimes 1_k & 0 \\ 0 & 0 & c \otimes 1_\ell \end{pmatrix}, \\ \rho_L^c(a, b, c) &= \begin{pmatrix} 1_p \otimes a & 0 \\ 0 & 1_k \otimes b \end{pmatrix}, \quad \rho_R^c(a, b, c) = \begin{pmatrix} 1_\ell \otimes a & 0 & 0 \\ 0 & 1_\ell \otimes a & 0 \\ 0 & 0 & 1_p \otimes b \end{pmatrix}, \end{aligned}$$

with possible complex conjugations here and there. The mass matrix is

$$\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & M_2 \otimes 1_k & 0 \\ 0 & 0 & M_3 \otimes 1_\ell \end{pmatrix}, \quad M_1, M_2 \in M_{p \times \ell}(\mathbb{C}), \quad M_3 \in M_{k \times p}(\mathbb{C}).$$



The fluctuations are

$$\begin{aligned} {}^fM_1 &= \sum_j r_j w_j M_1 \bar{v}_j^{-1}, \quad w_j \in U(\mathcal{A}_3), \quad v_j \in U(\mathcal{A}_2), \\ {}^fM_2 &= \sum_j r_j w_j M_2 \bar{v}_j^{-1}, \\ {}^fM_3 &= \sum_j r_j u_j M_3 w_j^{-1}, \quad u_j \in U(\mathcal{A}_1) \end{aligned}$$

and the action is

$$V(C_1, C_2, C_3) = 4k [\lambda \text{tr}(C_1 + C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1 + C_2)] + 4p [\lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3)].$$

The neutrino count and broken colour imply  $k = \ell = 1$  and  $p = 1$  or  $p = 2$ .

The case  $k = \ell = p = 1$  is obviously degenerate.

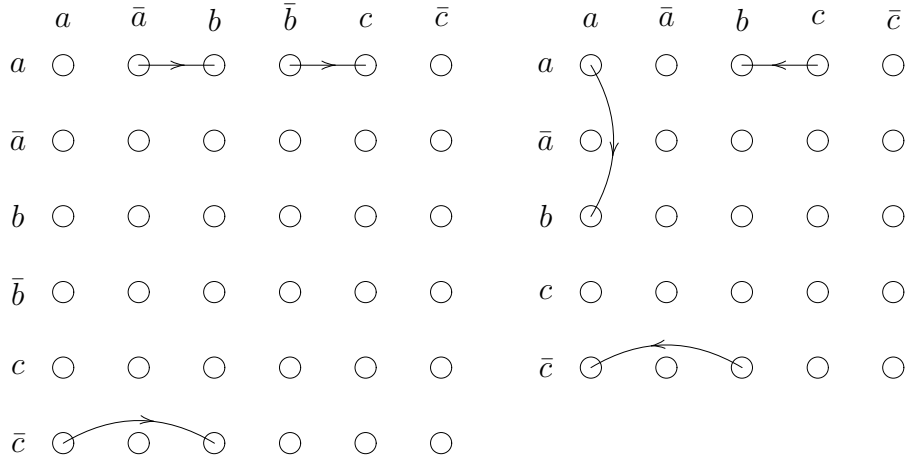
For  $k = \ell = 1, p = 2$  we have one neutrino.  ${}^fM_3$  fluctuates independently and can be pushed into the absolute minimum of the Higgs potential. Let us put  ${}^fM_1$  and  ${}^fM_2$  into one matrix

$${}^fM_{1,2} = \sum_j r_j w_j (M_1, M_2) \begin{pmatrix} \bar{v}_j^{-1} & 0 \\ 0 & \bar{v}_j^{-1} \end{pmatrix}$$

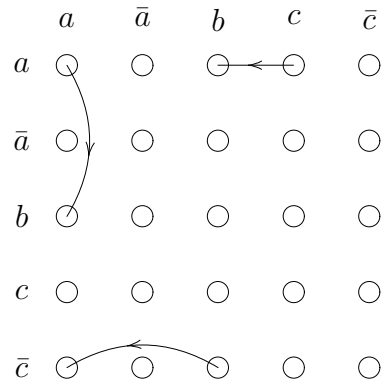
Since the  $\bar{v}_j^{-1} \in \mathbb{C}$  they commute with  $(M_1, M_2)$  and so

$${}^fM_{1,2} = C(M_1, M_2),$$

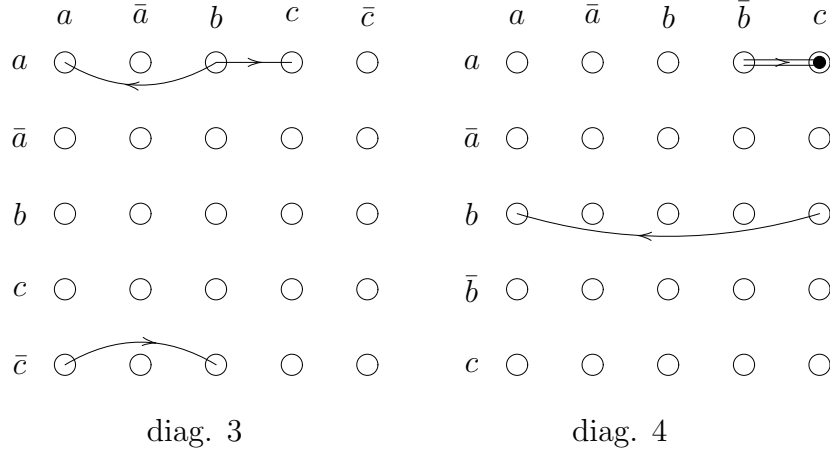
where  $C \in M_{2 \times 2}(\mathbb{C})$  is an arbitrary matrix.  $M_1$  has to be linearly independent of  $M_2$  because otherwise they would produce a second neutrino. It follows that  $(M_1, M_2)$  is invertible and we can choose  $C$  to be its inverse. In this way we reach the absolute minimum of the Higgs potential and the triple is degenerate.



diag. 1



diag. 2



## 5 Conclusion

The new models discovered with help of the computer complete the proof for up to three algebras given in [1]. We did not find anything of interest from the particle physics point of view but we gained confidence in our algorithm and it seems sensible to compute the case with four algebras.

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